# Perception of Direction and its Influence on Geometric Discoveries

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**Abstract**

This paper concerns two-dimensional images and how the mind processes them. It focuses on the domain of Euclidean geometry, and specifically on the perception and interpretation of direction. The ideas in this paper are currently being modeled using a computer program whose overarching purpose is the simulation of human geometric discovery.

**Keywords:** Geometric discovery; perception of direction; dynamic representation; conjecturing under uncertainty.

## A Simple Model for the Perception of Direction

### General Idea

It is a fact that some animals discriminate some directions. Polit and Bizzi (Polit & Bizzi, 1979) designed an experiment showing that monkeys (*macaca mulatta*) can reliably point in several distinct directions in the dark.

The ability to discriminate directions may be more sophisticated in humans than in other animals in the sense that humans are sensitive to a larger set of directions in the dark or purely in their minds without using pictures. (In this paper, the word "human" means, roughly, a healthy, normal adult.) It is not the purpose of this paper to say how many directions a human can discriminate, but to propose a plausible approximation.

For those readers reluctant to accept the existence of imagined figures, there is another approach. The following situation (which actually happened) is close to the ideas written in this paper. Two friends, say Ann and Ben, are talking about geometry. Ann is trying to show off a new proof she has found for a well-known theorem. However, she does not have a pen available, so while she talks, she "draws" imaginary figures in the air (such as a circle and a triangle) with her finger. From time to time, she points with her finger to specific parts of the figures in the air (such as sides of the triangle). Ben listens and watches, and he understands. This paper can be thought of as being about such figures drawn in the air.

### A First Type of Direction

We postulate that a number of directions are stored in long-term memory. If we use the hour hand of a clock to describe them, we can say that most people can "see" in their mind 1, 2, ..., 12. They also can "see" the 45-degree-related directions 1:30, 4:30, 7:30, and 10:30. In addition, they can imagine directions that are *close* to horizontal or vertical but are slightly off (in this paper they are called "tilted" and "steep"). There may also be sensitivity for yet other directions, but this first set is particularly salient. Another set of directions will be described in the next section.

People are most sensitive to the horizontal and vertical directions. They can very easily tell that an hour hand indicating 12:10 on a clock face with no numbers or other marks on it is not pointing at twelve o'clock, but they cannot tell that an hour hand pointing at 1:10 is not pointing at one o'clock. This discrepancy comes from evolution and from a long personal interaction with the real world, where a sense for the vertical and horizontal directions are very important for survival.

Perception of direction has some similarities to the perception of the sounds of a language. These similarities will be briefly discussed here for the sake of clarity and also to emphasize that the two types of imagination and perception — visual and auditory — are both brain processes and as such have similar underlying mechanisms.

A first analogy is between the fact that subjects have a small set of directions stored in long-term memory and the fact that speakers have a small set of phonemes stored in long-term memory. Sensitivity to vertical and horizontal directions is analogous to the fact that an English speaker is highly sensitive to the different sounds of the initial consonants in "the" and "da", whereas a Spanish speaker with no exposure to other languages is much less sensitive to this distinction, and it is quite likely that the two sounds will be indistinguishable.

Mathematicians unconsciously assume that they deal with an infinite number of directions, since they officially (i.e., professionally) show no preference for any direction. However, in an interesting way this is not true. Several geometry texts were inspected and it was found that, when the authors draw a "general" triangle (neither isosceles nor right), most of the time one side is horizontal. Another side usually makes approximately a 30-degree angle with respect to the horizontal side, and the third side makes approximately a 60-degree angle with respect to the horizontal side. (This shows a preference for directions like 1 o'clock and 10 o'clock.) For a simpler and perhaps stronger example, consider that a mathematician who is asked to draw a square on a blackboard will, with 99% probability, attempt to make two sides perfectly horizontal and two sides perfectly vertical. These observations support the claim about the saliency of the first set of directions mentioned above.

### Context-dependent Directions

There are more complex situations in everyday life than just pointing in a given direction. Imagine drawing a triangle one of whose angles is roughly 30 degrees and then bisecting the angle. The resulting line might point in a direction that is none of the clock directions mentioned above and yet it can easily be perceived or imagined (or drawn in the air). The direction of the resulting segment is perceived *within a context*, and the brain is able to re-interpret the new direction not as belonging to the small set of canonical directions mentioned above, but as something new. Following the analogy to language, this is similar to the perception of a mispronounced vowel within a sentence, something that happens often. For example, in the Spanish language, the phrase "no es grande" ("it is not big") is commonly (mis)pronounced close to "nuez grande" ("big nut"). Depending on the context, the brain will give it a meaning. Context-dependent directions (and in general, context-dependent objects) make the world of mental directions much richer. There are even content-dependent objects that can emerge from content-dependent objects. For example, a bisector together with one side of the original angle can be used to define another bisector.

### Clouds of Geometric Objects

We will define a "cloud" as a plausible region where a human imagines some geometric entity to be located. For example, a subject who is given a blank piece of paper and asked to draw a point at the center will almost certainly draw a point close to the center. However, before the subject draws the point, they first have to imagine it in their mind. This implies that they need to set a context (or scale) in their mind, which is linked to the dimensions of the piece of paper. If this activity is performed several times, the resulting set of points will make up a cloud whose densest area is close to the center of the piece of paper. The cloud's size will be small in comparison to the piece of paper.

Just as a point has a cloud, so a direction has a cloud as well. Thus, a line passing through a point inherits the clouds of the point and the intended direction. A random triangle has a cloud, too, which is inherited from its three vertices and three sides.

A point is in the cloud of another point if the imaginer is not sure that the two points are not the same. In this context, "clouds" in language are abundant. For example, the sound of the word "Mary" might lie in the cloud of the word "marry" for many individuals.

The reason for the existence of clouds of this sort is that directions in the brain are the result of the competition of cells that have been excited during a stimulus. In fact, in the presence of a stimulus, there is activity of populations of cells corresponding to a small set of specific directions (Georgopoulos, 1988). In our particular domain, the stimulus is a mental figure or even a drawing in the air. The mechanism underlying this competition is analogous to the phenomenon of *lateral inhibition* in the auditory system (suppression of firing by nearby cells that have greater excitation).

### The Conscious and the Subconscious

The mathematician Jacques Hadamard devoted considerable time to understanding the psychological processes in mathematical discovery, and among his conclusions was the idea that the object that is the focus of the mind at a given moment is at the level of consciousness, while other objects are at lower levels. Hadamard admitted that there are serious difficulties in defining such levels, but we can assume that there are two levels for the purpose of developing a simplified computational model. One level is conscious and the other is unconscious. A good example for this simplified model is the idea that the current sentence in a conversation is at the conscious level of the speaker, while the just-uttered sentence is at the subconscious level.

In processing images, the current object, or the object that at a given time is the focus of attention, is at the conscious level. Other parts of the mental image are at the subconscious level. For example, if a human is asked to imagine a triangle, the image of three line segments goes to their conscious level. If, shortly thereafter, they are asked to imagine drawing the bisector of one of its angles, the triangle as a whole fades away (it goes to the subconscious level) while the chosen angle rises to the conscious level. It is at this instant that the bisector can be "drawn" in the mind.

It is problematic to ask how many objects at a given time can reside at the conscious level. Psychologists typically say "$7 \pm 2$." Imagining a triangle requires one to imagine several objects — three sides, three vertices, and three angles. However, one of the powers of the mind is to *group* objects. The theory behind our model presumes that when we think "triangle," we are, in a certain sense, imagining a unitary object. If, however, we need to focus on a part of this triangle such as a vertex, then we bring the vertex to the conscious level while the triangle as a large whole descends to the subconscious level. At the subconscious level, we of course still know the triangle is there, but we are no longer imagining it or "seeing" it. This is in agreement with the Necker-cube phenomenon, whereby two alternative images swap roles, one being conscious and the other inaccessible in the unconscious.

## A Glance at the Computer Model

### Interface

The computer program called *George* is intended to model geometric discovery in pitch darkness, simulating the behavior of a curious human (a numerical parameter can be set to give George different levels of curiosity). George has a structure that imitates the *mental plane* — that is, the system in the brain that imagines figures and their parts, and is able to quickly shift attention from one part to another part (Lara-Dammer, 2007). One of its features is the "mental" manipulation of geometric entities such as lines and triangles. Another feature (more germane to this paper) is the handling of direction. George is still under development and is not yet ready for geometrical discovery on its own, but at its current state already possesses some desirable psychological traits, such as the dynamic representation of geometric entities (points, triangles, etc.), and the manipulation of these entities at two levels (conscious and subconscious).

As part of its human interface, George has a text field in which a person can ask the program to imagine geometric figures. It also has a text area to which it sends responses. In addition, it possesses a drawing area on which representations of geometric entities are drawn.

Images in the mental plane (or figures drawn in the air) are not trivial to represent on a computer screen. They are not like static figures drawn on a piece of paper. There are fundamental aspects that need to be addressed. One aspect is that an imagined object is dynamic (e.g., it is subject to certain simple commands such as "move" or "erase") and exists only for some short length of time at the conscious level. An imagined object may still exist even though is not currently "visible" (i.e., it has descended to the subconscious level). If we wish to observe and interpret the behavior of a computer model of imagined figures, we need a way to represent its imagery on the screen.

To this end, George's mental figures have been assigned visual encodings. For example, a segment at the conscious level is represented as a solid line, and when is at the subconscious level, it is dashed. A point at the conscious level has a bigger size than when it is at the subconscious level. A triangle at the conscious level is painted in its interior and when it is at the subconscious level it is not. (These choices for encoding are likely to change with the evolution of the program.)

Another aspect that a model has to address is the transition of a figure from the subconscious level to the conscious level, and vice versa. When an image appears at the conscious level, it is not exactly the same as it was during a previous apparition, even though both images correspond to the same mental figure. It is only a plausible image that is close in location, size, and orientation to the original, but is not the same image. Put in a different way, the old and the new images are in the cloud of an ideal or platonic figure.

Suppose, for example, that Ann and Ben are involved in a dialogue and at a given time they are thinking of a triangle and a bisector of one of its angles. In their minds, while they are performing the mental operations to "draw" this bisector, some parts of the triangle fade off. As is common in conversations, Ann unconsciously "draws" the triangle and its bisector in the air with her index finger. At this precise instant, however, she is interrupted by a phone call. After attending to her phone call, she resumes paying attention to the image, pointing to its parts, but of course the image is different from the image she originally drew in the air. It might even be in a different room, of a different size, and so on. However, she assumes unconsciously that the figure is the same.

In the computer model, the triangle is initially painted, but after it descends to the subconscious level, the angle to be bisected ascends to the conscious level, and when this happens, the triangle ceases to be painted, and the angle is highlighted along with its bisector. The sides of the triangle remain dashed throughout this process since they were never brought to the level of conscious attention.

George has a useful command that allows a person to get a sense for the cloud of an imagined figure. If this command is applied to some point in an imagined figure, the point will not appear in a fixed spot but will instead move about at random in a limited region of the screen.

The cloud of a horizontal line is encoded on the screen by a line that is almost perfectly horizontal but that floats up and down a very small amount. In contrast, the cloud of a 1 o'clock line through the center of the screen does not have nearly as fixed a direction. Its slope changes rapidly (though not drastically) at random, and at the same time the line's location also floats a little bit. This aspect of the model is intended to reflect the lower sensitivity of humans to a 30-degree slope, as was discussed earlier.

The cloud of a standard triangle has a horizontal base that floats slightly up and down. The left and right ends of this line jiggle randomly mostly from left to right and back, whereas the third vertex of the triangle jiggles randomly in all directions. Naturally, this contrast among vertices is intended to reflect human beings' greater sensitivity to the horizontal direction.

The idea of the cloud of an image has some similarity to the idea of "electron cloud." The natural human intuition that the orbit of an electron can be thought of as a shape that can be perceived by our eyes (as a ball moving on a parabola) is wrong. We cannot locate the electron in a deterministic way. In much the same way, we can never draw the same triangle in the air.

## A Model of Mathematical Conjecturing under Uncertainty

Each figure is associated with a cloud. Therefore, if George is asked to imagine a point in the center of its mental plane, the image is manifested to the outside world as a point close to the center. The point is not merely a pair of coordinates as it would be in a purely geometric program. If the same question is proposed at a different time, the position of the second point is likely to be close to the first but not exactly in the same position. Moreover, if the point descends to the subconscious level and later rises to the conscious level, the *mental image* for George will be the same, but its *representation* to human observers will not be quite the same (as in the example of Ann and Ben's dialogue).

In general, George has a rough sense of location but it does not know exactly where a figure is located. The intersection of two bisectors of a triangle is, for George, a point somewhere in the interior of the triangle. It is quite vague *where*, however. If a second pair of bisectors is drawn, the resulting point is not necessarily the same for George as the first, although George has a way to detect that they are probably close. (All three bisectors of any triangle do in fact meet in a single point called the "incenter," but George does not know this theorem.)

For more complicated geometrical situations, humans normally refresh their mental image by "re-thinking" the object (or re-drawing it in the air), to reassure themselves that the points are in fact close. A problem, though, is that when one does so, the mental image of the triangle is subject to small variations of which one is not aware during one's unconscious processing.

The art of geometrical conjecture starts at the instant in which a large situation created out of unrelated pieces (such as the three vertices of a triangle) presents seeming or possible regularities or patterns. Examples of such patterns are when three apparently unrelated lines seem to meet in one point, or three apparently random points seem to lie on one line, or when a circle seems to be tangent to a line.

Some humans are surprised by and attracted to these kinds of unexpected regularities in an irregular world, and some are not. The degree of surprise and attraction varies from person to person. This parameter defines the level of motivation and curiosity for a human to explore a geometric figure.

In order to make a conjecture, a human has to be sensitive to and interested in patterns. Otherwise, it would be analogous to listening to a lecture in a foreign language. The mind will not be able to find any meaning or any pattern. The art of conjecturing requires another set of mechanisms that allows a person to jump to conclusions on the basis of some evidence. The next section discusses how George simulates the ability to make simple intuitive geometric conjectures.

## A Pathway to the Discovery of the Incenter

Let us consider a human who does not know of the incenter, such as a twelve-year-old not yet exposed to geometry.

To simulate such a human, we would ask George to imagine a triangle and then to draw the three bisectors, one at a time. When imagining this triangle, George does what almost any human does. One side would be "drawn" horizontal, and the other two sides would point to two salient directions, such as 1 o'clock and 10:30. Drawing the bisector of the first angle requires one to send the triangle to the subconscious level and bring the chosen angle to the conscious level. Drawing the bisector of the second angle requires sending the first angle and its bisector to the subconscious level, and bringing the second angle to the conscious level. The third bisector is drawn in a manner similar to the second.

At this point we ask George to *intersect* the first two bisectors. This operation requires George to send the third angle and its bisector to the subconscious level and to "resuscitate" the first and second bisectors. (A human can do this in two ways — one by just remembering and the other by redrawing.) A point emerges from this intersection. The emergence of this point means that the point itself resides on the conscious level while the other parts of the figure relinquish the conscious level, descending to the subconscious level. Now we ask the program to intersect the second and third bisector. This causes the newly emerged point to descend to the subconscious level and the corresponding bisectors to ascend to the conscious level. At this moment, a second point emerges, just as did the first point. And then, much as in the previous step, a third point emerges.

Now we ask the program to observe its subconscious. Maybe there is a pattern. At the current state of the program, it can observe a very limited number of patterns, so we explicitly ask the program to *group* any points that are close enough. (A later version of George will think of the idea of grouping on its own, when there is sufficient reason to do so, but at this stage that is beyond George's powers.) When points are mentally close enough, this fact triggers a state of curiosity which "wonders" if the three distinct points are actually just one point.

In reality, of course, this step is not deterministic. Different human subjects, depending on their personal levels of curiosity, will group objects for different reasons, one of which will involve their intuited proximity in the mental plane. This is the way that George is planned to work.

George triggers a heuristic for conjecturing that distinct points are *not* in fact distinct whenever the points can potentially be in the cloud of a single point. An analogous situation to this in language is the following. We hear a phrase which sounds identical to another phrase (for instance, "till late" and "till eight"; "the mall and "them all"; "this guy" and "the sky"). Most of the time we can disambiguate the utterance by context. However, there are rare cases in which the context is not sufficient for disambiguation because both meanings of the utterance fit well in the context. A concrete example follows. One of the expressions of a poetic song was heard by F.L. as "solo para serle" ("only to be something to her"), but this sounds nearly identical to "solo para hacerle" ("only to make something to her"). Both lines made sense in the context of the song. Unable to know for sure what the song's author really intended, F.L. entered a state of curiosity and felt compelled to ask other people what they thought. This did not help, however, since half of the interviewees heard it one way and the other half heard it the other way. Interestingly, the published lyrics of the song did not help either, because half of them had it one way and the other half had it the other way. Twenty years later, F.L. still does not know what the original poem said.

This type of event does not occur frequently, but when it does occur, it can provoke strong reactions. In the case of language, phonetic blurs with dual meanings usually invite us to laugh, and jokes and puns are created around them. In the case of mathematical discovery, such blurs or conjectured coincidences can generate excitement and thrill, at least in the mind of someone whose "curiosity" and "surprisability" parameters are tuned high.

A state of curiosity is crucial to discovery. It is a state in which we feel a need to act in order to resolve our uncertainty. To satisfy this type of curiosity, George (whose mental plane is very imprecise and blurry) has a very precise tool named "Tricycle". This is a dynamic geometry program (similar to the commercially available program *Geometer's Sketchpad*). (Hofstadter (Hofstadter, 1997) describes the process of geometric conjecturing with the help of such a precise computer program.) With the help of this "assistant", George (actually, any human too) is able to draw high-precision figures, which give far more confidence. (The physicist Richard Feynman once claimed (Feynman, 1985) that his geometric mental imagery was so sharp and precise that, in effect, had a built-in personal "Tricycle." If this is true, Feynman's mind was extremely exceptional. More likely is that he was exaggerating, as he often did for the sake of entertaining his listeners of readers.) In the case of the three intersection points of the bisectors, they will all appear at a single point. This is not a point of the mental plane, of course, but a genuine point in a *Euclidean* plane.

Some people may feel that this example of the three bisectors meeting at a single point is too obvious, but this is almost certainly the result of having learned at some stage in school that a triangle's three bisectors meet in a single point. In this paper, we have used several paragraphs to describe this seemingly simple conjecturing process. This shows that this discovery is in fact rather subtle, and is not a trivial matter at all. After all, a triangle is determined by three arbitrary points not related in any way. Why on earth should its three bisectors all converge at a single point? It is a kind of miracle — at least to *some* minds it is.

We conclude with a quote by Albert Einstein (Einstein, 1983), which suggests that we give greater respect to "simple" geometric conjecturing:

> At the age of twelve, I experienced a wonder in a booklet dealing with Euclidean plane geometry, which came into my hands at the beginning of a school year. Here were assertions, as for example the intersection of the three altitudes of a triangle in one point, which — though by no means evident — could nevertheless be proved with such certainty that any doubt appeared to be out of the question. This lucidity and certainty made an indescribable impression on me.


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