A strangely symmetric pattern
involving conjugacies and "local" and "global" bisectors

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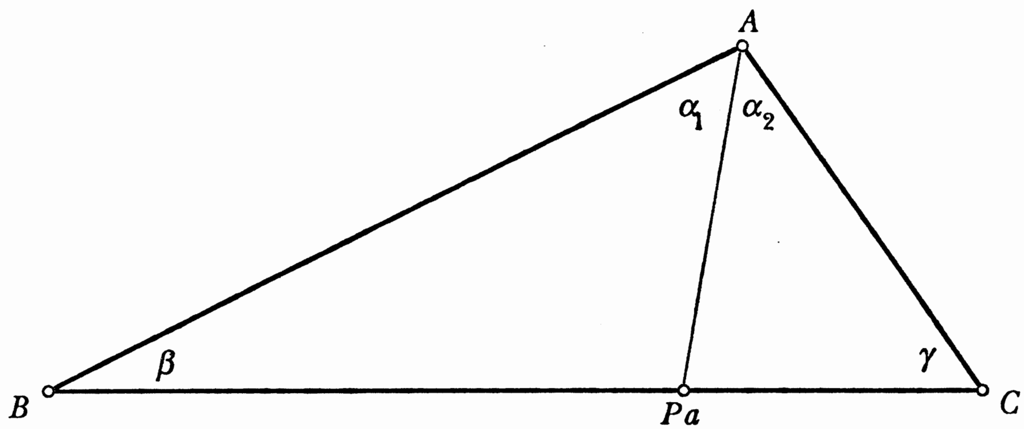
I recently made a geometrical finding that is more of a *pattern* connecting certain facts than a mathematical theorem (perhaps there is a theorem lurking behind it). It involves the interrelationships of certain special points and conjugacies in a triangle.

We begin with the fact that many special points come from *bisections* of one sort or another: the centroid (involving the midpoints of sides), the circumcenter (perpendicular bisectors of sides), the incenter (angle bisectors). All of these bisections involve *local* operations: the bisection of an angle regardless of the other angles of the triangle, or the bisection of a segment without regard to its neighboring segments. This led me to ask: What about *global* types of bisection? Thus, instead of bisecting the *side* opposite a vertex, why not bisect the entire *perimeter*? It is well known that the perimeter-bisecting point is the point of tangency of the excircle across from the given vertex. So that's an example of a *global* type of bisection.

Another example would involve, instead of bisecting the angle at a given vertex, angularly bisecting the entire triangle, by which I mean finding a cevian such that it divides the total angle-sum of the triangle into two equal parts, which of course must both be 90°. Here is a picture of what I mean:

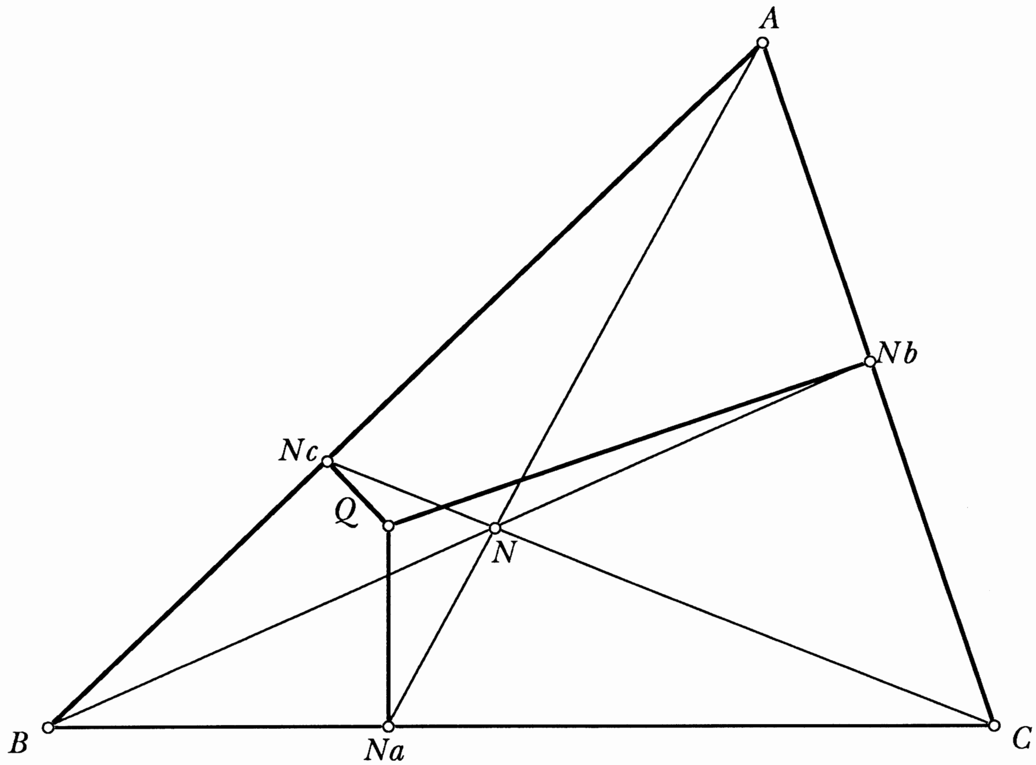


We want segment A–*Pa* to be such that $\alpha_1 + \beta = \alpha_2 + \gamma = 90°$. Pretty obviously, this will happen only if we are dealing with an altitude. So that's what global angle bisection yields.

We've now seen two types of global bisection — one based on lengths, one based on angles. Now we utilize these notions to define a couple of new *centers,* as well as in classifying centers both new and old.

What happens, for instance, if you find the meeting-point of the cevians to the global length-bisectors (*i.e.,* the perimeter-bisectors)? This, by definition, is ABC's Nagel point N (see the picture below). So this is a way of characterizing the Nagel point in terms of a global bisection operation.

And what if we consider the *perpendiculars,* rather than the cevians, erected at these global length-bisection points? Do they meet in a point? It happens that they do — namely, the point that I call "Q", also shown below (here, the N's are the perimeter-bisectors, or equivalently, the points of tangency of the three excircles):



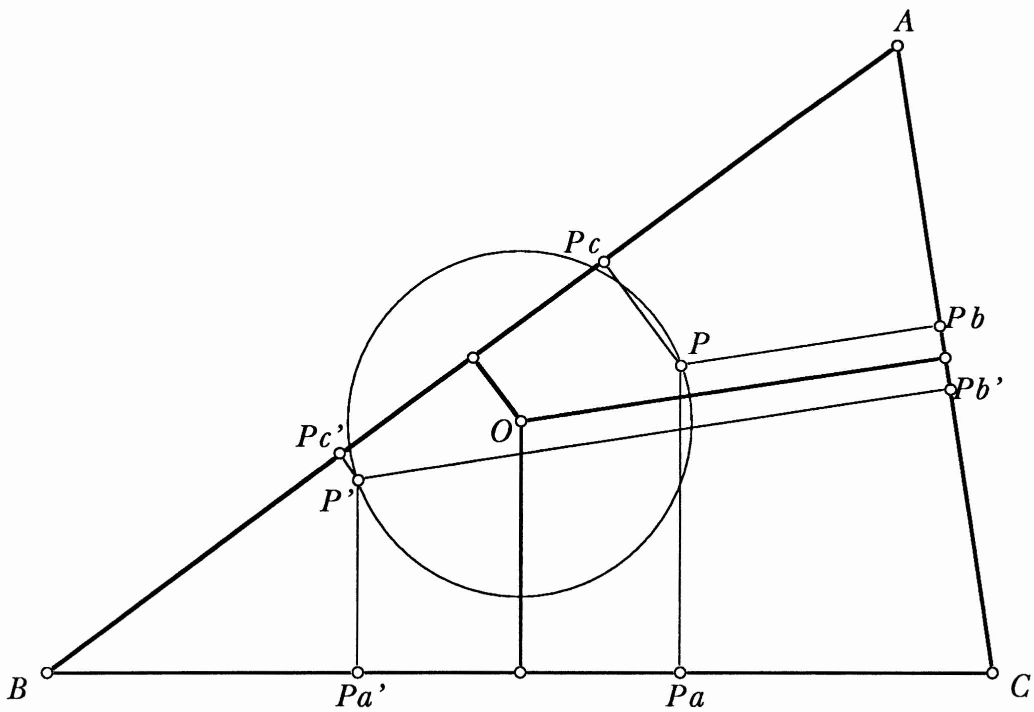
We conclude our quest for centers based on global bisectors by asking what the meeting-point (if any) of the three global *angle* bisectors is. Since these lines are none other than the altitudes, the answer is trivially obvious: the orthocenter H.

So now we can make a little table:

		Local	Global
Angle bisection	using cevians...	*I*	*H*
Length bisection	using cevians...	*G*	*N*
	using perpendiculars...	*O*	*Q*

From an esthetic and analogical-thought point of view, it would seem that angular bisection should have not one but two rows, one involving cevians and the other one somehow involving perpendiculars. The problem is that *a priori,* it's not clear what it would mean to carry out angle bisection using perpendiculars, or if the entire notion even makes any sense. (It turns out that one can find some sense in the idea, as I shall show below.)

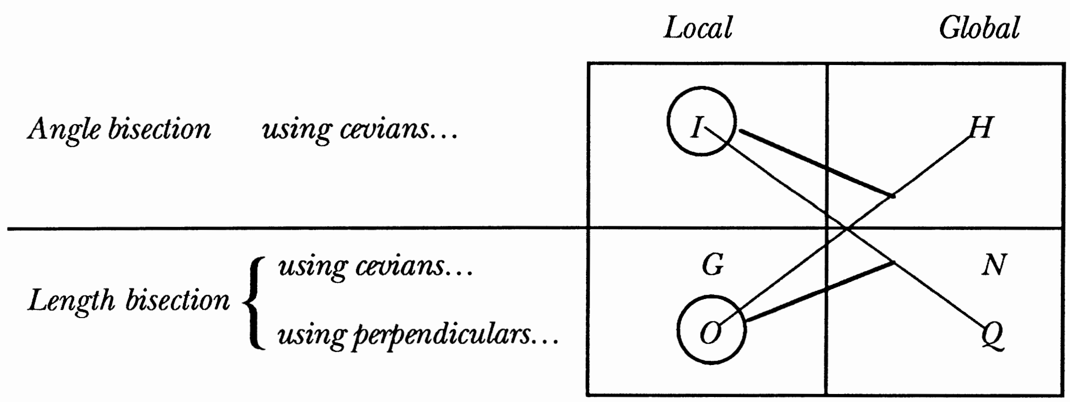
I now wish to show some relations of this table to conjugacies, but before I do so, I need to define one new kind of conjugacy. The idea is very close to isotomic conjugacy. What you do, given point P, is first to drop perpendiculars from P to the three sides. Then reflect each perpendicular line across the perpendicular bisector of the side in question. This gives three new perpendiculars, which turn out to all meet in a new point P'. As is nearly obvious when one thinks about it a moment, P' is just the reflection of point P through the circumcenter O. Here is a picture of the operation:



Although this "circumcentric conjugacy" seems almost trivial, its definition has the "feel" or "style" of other conjugacies, so let us treat it as a perfectly good one.

One interesting but simple fact about circumcentric conjugacy is that, since the tangency points of the incircle and those of the excircles are located at equal distances from the sides' midpoints, the incenter I and the point Q defined earlier (the place where the perpendiculars at the excircles' tangency points all cross each other) are each other's conjugates in this new sense.

Let us look at our little six-entry table of special points, with conjugacies in mind.

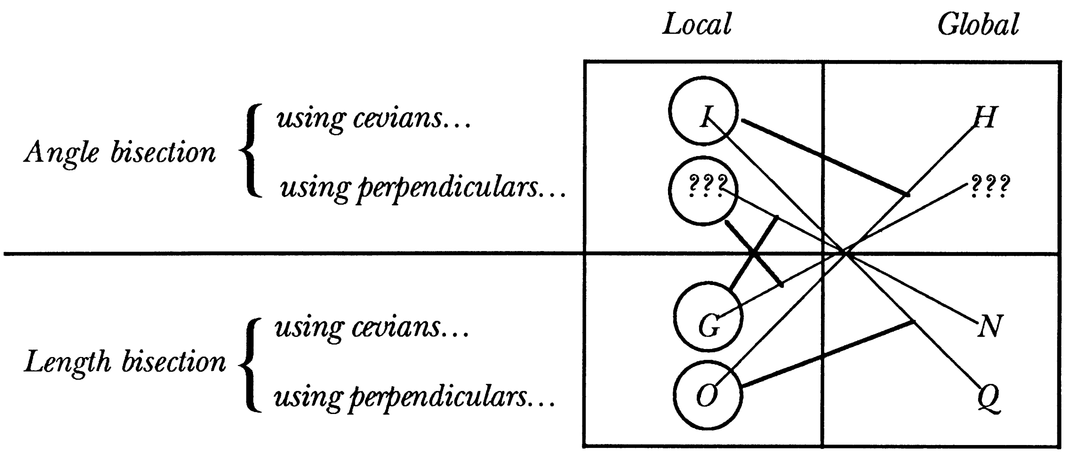


Under the conjugacy whose fixed point is I — namely, *isogonic* conjugacy — the points H and O are each other's conjugates. And under the conjugacy whose fixed point is O — namely, *circumcentric* conjugacy — the points I and Q are each other's conjugates. These facts are symbolized by the thin lines connecting letters (showing pairs of points that are conjugates), and by the slightly heavier lines that effectively *label* the thin lines, by linking them circled letters (showing which conjugacy is involved — "I" meaning isogonic, "O" meaning circumcentric).

Now in some ways this is a pleasing diagram, but it might leave one a bit unsatisfied, given that length bisection comes in two varieties (cevian and perpendicular) whereas angle bisection seems to come in only one. Moreover, only the top and bottom rows are involved in the conjugacy connections, while the middle row is left out of the picture. This would seem to suggest that there might be a missing row, and that we should try to fill it in.

(Incidentally, this strong desire to complete what feels like an incomplete pattern of symbols, and to do so purely by using analogy and symmetry considerations, reminds me very much of what Clerk Maxwell is reputed to have done in adding the famous displacement-current term to one of his four equations of electromagnetism (actually, he had *eight* equations, because two of them were scalar equations and two of them were vector equations, and he wrote out each vector equation as three equations — one for each coordinate). The story is probably somewhat exaggerated, but still it sticks in one's mind. Similar discoveries obtained by filling in missing elements of patterns were made by elementary-particle physicists who took Gell-Mann's eightfold-way diagrams and hypothesized that where there were blanks in them, there simply *had* to be particles, and lo! they went out and found them. Even today, this kind of search for particles is going on — there is a powerful belief in symmetry that argues that there *must* exist a "top quark", even though it hasn't been seen, and physicists are so confident in this that they speak nonchalantly of the "six quarks" in the "standard model", despite the fact that as of yet, there is *no experimental evidence at all* of such a quark.)

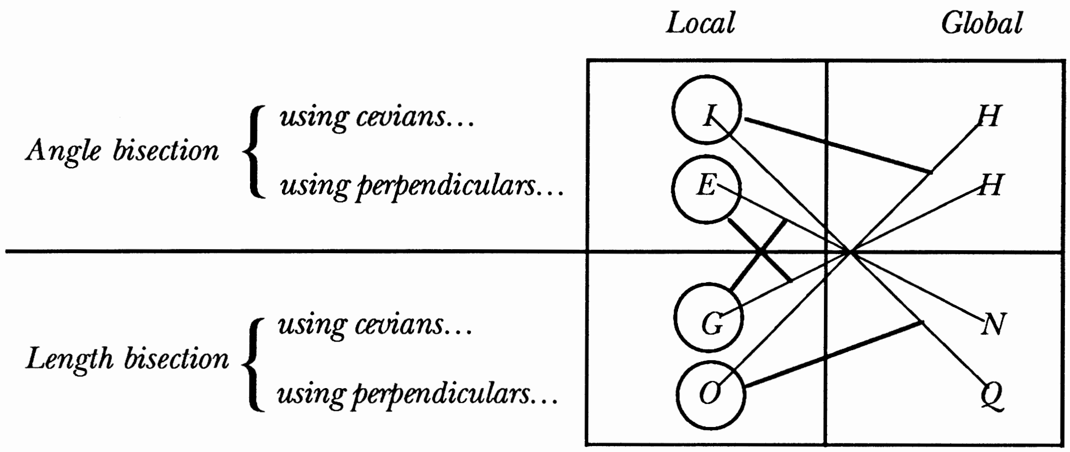
But back to the matrix. Here is how a more complete and more symmetric picture would look:



The question is, naturally: What do the question marks stand for? Well, one thing that helps is that we know a conjugacy whose fixed point is G — namely, *isotomic* conjugacy. According to the picture, then, N and the mystery item in the left column should be isotomic conjugates. But it's well known that the isotomic conjugate of the Nagel point N is the Gergonne point, E, for the same reason as was mentioned above: on any side, the *incircle's* tangency point and the *excircle's* tangency point are mirror images with respect to that side's midpoint. So this means we can fill in one of the blanks with the letter "E".

Now that we know that "E" goes in the lefthand column and second row, we can try to interpret the Gergonne point as a center whose definition involves both local angle bisection and perpendiculars somehow. This is not hard to do. In fact, we can do it as follows: Given ABC, go to the point where all three of its local angle bisectors meet, then drop perpendiculars to the sides and use those new points to define three new cevians, which will meet in a point. (This is how the notion of "global angle bisector" can be made to make sense.)

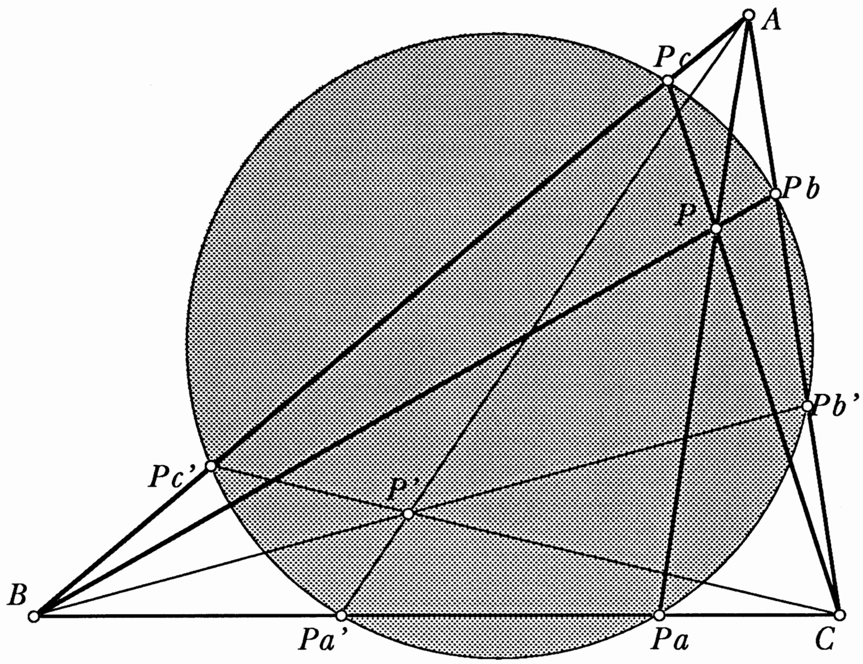
So to get the mystery item in the righthand column, we need merely "globalize" this characterization of the Gergonne point E. To do so, we simply replace the phrase "local angle bisectors" by the phrase "global angle bisectors" and carry out the recipe: go to the point where all three *global* angle bisectors meet, drop perpendiculars to the sides, and use those new points to define three new cevians; hopefully, then, these cevians will all meet in a point. Well, the point where all three *global* angle bisectors meet is the orthocenter H. If you drop perpendiculars and then draw cevians to the feet, you bounce right back to H again. Thus our recipe suggests that the second missing item in the matrix is H once again. So here is how the completed matrix looks:



The only thing we haven't yet figured out is whether there really exists a conjugacy whose fixed point is E, the Gergonne point, and which carries the centroid G into the orthocenter H, and vice versa.

Now upon completing this matrix, I looked high and low for such a conjugacy. I came up with one candidate conjugacy after another, each one based on principles of esthetics that I thought, at the time, were impeccable and certain to work. Each time, I made my construction using Geometer's Sketchpad, and each time, it looked like it worked at first, but then, as I moved things around at first, it started to fail, in the sense that three cevians or perpendiculars that had appeared to always meet in a single point would start to diverge slightly, and then, as I continued to move my test point around more, they would diverge even more, and my heart would sink. Seriously, or at least semi-seriously, I would ask myself, "How could such a beautiful idea not work out? How could Nature have missed this opportunity for a beautiful relationship? It seems that Nature made a mistake!" And I would be very sad. And then I would make another attempt. Eventually, I lost heart and quit the search, at least for a while.

Then one day, thinking about a completely different problem, I was looking at the nine-point circle and thinking about how it cuts each of the triangle's sides in two incredibly important points (a midpoint and an altitude foot), and it occurred to me that the fact that there is a circle passing through them makes those two points in some sense complementary to each other. So it was but a little leap to the idea that I might define a conjugacy using this kind of complementarity of two points on a side. So my new candidate conjugacy was this: Take any point P, and pass cevians through it. These lines hit the given triangle's sides in three points *Pa, Pb,* and *Pc.* Make the circle that passes through these three points. It will in general pass one more time through each side, thus defining three new complementary points: *Pa', Pb',* and *Pc'.* Make cevians through these new points; they should hopefully meet in a single point, the conjugate point of P. Here's a picture:



It does indeed work in the sense that P' does exist for every P, and so I was very pleased with my discovery. But mind you, I was not thinking of it as a solution to the previously sought conjugacy. It was just a pleasing new conjugacy, period.

But then the next day, in the car, I was musing about this new conjugacy and I asked myself, "And what is its fixed point?" I realized that this must occur when the circle involved is tangent to all three sides, which means it is the incircle. At first I thought, "So it's the incenter!" — but in a moment this erroneous thought was dislodged by the correct answer: the Gergonne point E! And as soon as I realized this, I remembered that the nine-point circle tells me that G and H are indeed each other's conjugates, and so here precisely was my hoped-for conjugacy, coming in through the back door! This discovery strongly suggested that my little matrix was revealing something very systematic about some behind-the-scenes interconnections between special points and conjugacies.

I must admit, though, I don't know exactly what the existence of this matrix tells me. It seems very pretty in itself, in the way it unifies four diverse conjugacies, but one wonders if there isn't more to the story — if there isn't a single reason for it all, behind the scenes. And perhaps there are further conjugacies and special points that might be addable into this pattern, in some way.